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An improved slender-body theory for Stokes flow

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The present study examines the flow past slender bodies possessing finite centre-line curvature in a viscous, incompressible fluid without any appreciable inertia effects. We consider slender bodies having arbitrary centre-line configurations, circular transverse cross-sections, and longitudinal cross-sections which are approximately elliptic close to the body ends (i.e. prolate-spheroidal body ends). The no-slip boundary condition on the body surface is satisfied, using a convenient stepwise procedure, to higher orders in the slenderness parameter (ϵ) than has previously been possible. In fact, the boundary condition is satisfied up to an error term of $O(\epsilon^2)$ by distributing appropriate stokeslets, potential doublets, rotlets, sources, stresslets and quadrupoles on the body centre-line. The methods used here produce an integral equation valid along the entire body length, including the ends, whose solution determines the stokeslet strength or equivalently the force per unit length up to a term of $O(\epsilon^2)$. The $O(\epsilon^2)$ correction to the stokeslet strength is also found. The theory is used to examine the motion of a partial torus and a helix of finite length. For helical bodies comparisons are made between the present theory and the resistive-force theory using the force coefficients of Gray & Hancock and Lighthill. For the motion considered the Gray & Hancock force coefficients generally underestimate the force per unit length, whereas Lighthill's coefficients provide good agreement except in the vicinity of the body ends.

1. Introduction

The singularity method for Stokes flow has been shown by Chwang & Wu (1974, 1975) and Chwang (1975) to be a very useful tool in constructing solutions to a wide variety of flows past axisymmetric bodies which need not be slender. Here we consider the flow past a slender body of circular transverse cross-section and arbitrary centre-line configuration having a prescribed centre-line motion. The solution is constructed by distributing appropriate singularities along the body centre-line using the previously mentioned works as a guide in selecting the needed singularities. Attention is restricted to bodies whose longitudinal cross-section shape is elliptic in the neighbourhood of the body ends (i.e. prolate-spheroidal ends). For a slender body we require the slenderness parameter, $\epsilon = b/l$ (where b is a typical transverse cross-sectional radius and l the body half-length), to be small compared to unity. In recent years the slender-body problem has received considerable attention (Tuck 1964; Cox 1970; Tillett 1970; Batchelor 1970; Keller & Rubinow 1976). Those studies considering curved centre-line configurations (Cox 1970; Keller & Rubinow 1976) have used the method of matched asymptotic expansions to generate solutions valid away from the body ends. Cox expands the solution directly in powers of $1/\ln\epsilon$, whereas Keller & Rubinow first obtain an integral equation representation of the solution which they solve using an

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iteration scheme similarly producing an expansion in terms of $1/\ln\epsilon$. Only Cox has applied the method to examples involving bodies having curved centre-lines. In such applications it is generally only practical to obtain the force per unit length neglecting terms of $O[(1/\ln\epsilon)^3]$. It is worth pointing out that in the method of matched asymptotic expansions not only is the end effect neglected as an approximation, but also the effects of local centre-line curvature. In those methods the inner expansion of the velocity field is taken to be the flow past an infinitely long straight circular cylinder. The present study retains both the end effect and the effect of the local body curvature. We do, however, limit our investigation to slender bodies whose centre-line radius of curvature is everywhere large compared to the cross-sectional radius b and slender bodies whose centre-line does not reapproach itself. In the latter case, for example, we must omit a helix of high pitch or a nearly closed partial torus.

In the present method the velocity field is constructed by superposing distributions of stokeslets, doublets, rotlets, sources, stresslets and quadrupoles on the body centre-line with initially unknown strengths. This integral representation of the velocity field is then partially integrated by constructing expansions to the integrand valid for points close to the body surface (the expansions being valid along the entire body length, including the ends). Satisfying the no-slip boundary condition neglecting terms of $O(\epsilon^2\ln\epsilon)$ results in an integral equation, valid for the entire body length, which determines the stokeslet strength (i.e. the force per unit length) within an error of $O(\epsilon^2)$. The other singularity strengths are found as functions of the stokeslet strength. Proceeding to one further term in the boundary condition produces the $O(\epsilon^2)$ correction to the stokeslet strength. Satisfying the boundary condition with the same degree of accuracy using an expansion in powers of $1/\ln\epsilon$ requires retaining an infinite number of terms in the expansion – the inherent disadvantage in using such slowly convergent expansions. The utility and accuracy of the method for slender bodies presented here was demonstrated by Johnson & Wu (1979) for the flow past a slender torus. For that special case they were able to calculate the force per unit length on the torus in closed form neglecting terms of $O(\epsilon^2)$.

We begin the study with a general discussion of the boundary-value problem considered. This is then followed in §3 with the special case of a slender body whose longitudinal cross-section takes the form of an ellipse over the entire body length when the body centre-line is straightened (i.e. a slender prolate spheroid having a curved centre-line). In §4 the extension of the methods to bodies whose longitudinal cross-sections need only be elliptical in the vicinity of the ends is given. Finally, in §5 we apply the methods developed to a few simple motions involving a partial torus and a helix. Comparisons are made with existing results, including the force coefficients of Lighthill (1976) for a helix.

2. General problem

The governing equations and boundary conditions for the Stokes flow problem considered are

$$\left.\begin{aligned}&\nabla p = \mu\nabla^2\mathbf{u}, \quad \nabla.\mathbf{u} = 0,\\ &\mathbf{u} = \mathbf{U}(\mathbf{x},t) \quad (\mathbf{x} \text{ on the body surface } S_b),\\ &\left.\begin{aligned}\mathbf{u}\to 0\\ p\to 0\end{aligned}\right\} \quad \text{as} \quad \mathbf{x}\to\infty,\end{aligned}\right\} \tag{1}$$

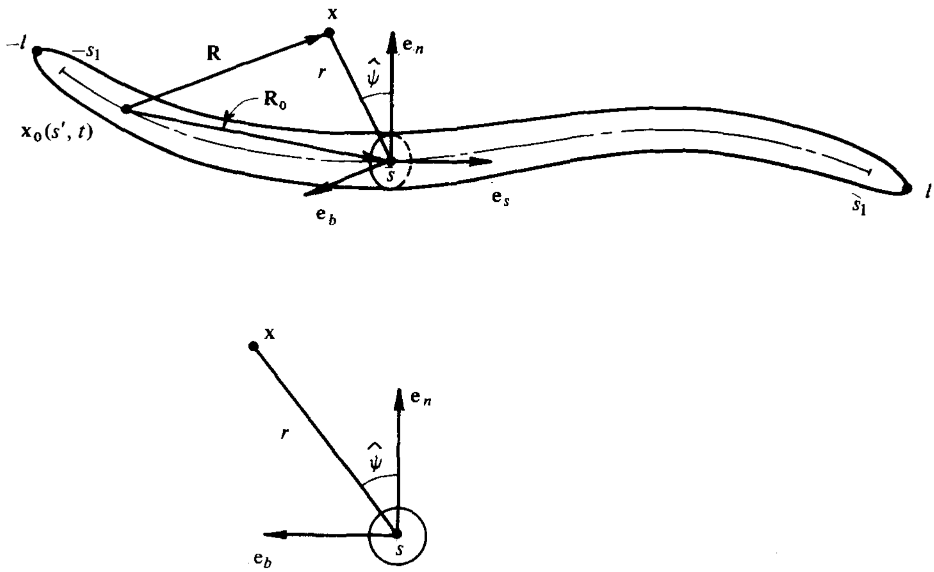

FIGURE 1. Slender-body geometry ($\hat{\psi} = \pi - \psi$).

where $\mathbf{u}$ is the velocity vector, p the pressure, μ the constant viscosity coefficient, t the time, and $\mathbf{x}$ the position vector in three-dimensional Euclidean space. We note that an additional undisturbed flow at infinity which satisfies the Stokes equations could be readily included by appropriately redefining the velocity in order to obtain equations in the same form as (1). The linearity of the governing equations allows us to construct solutions to a given problem by superposing the fundamental singular solutions of the Stokes-flow equations.

We will consider the case of arbitrary motion of a flexible slender body with circular transverse cross-section and length $2l$. The centre-line configuration of the slender-body motion can be generally prescribed in the parametric form

$$\mathbf{x} = \mathbf{x}_0(s,t) \quad (-l \leqslant s \leqslant l), \tag{2}$$

where s is the arc length along the centre-line. The parametric function $\mathbf{x}_0(s,t)$ is assumed at least twice differentiable in s and t. The body centre-line can in general be extensible. Therefore for arbitrary t we prescribe the arc length position of a material point, which was originally at s_0 for $t = 0$, by the following mapping from s_0 to s,

$$s = s(s_0, t) \quad \text{where} \quad s(s_0, 0) = s_0.$$

In the following, all quantities having the dimensions of length will be non-dimensionalized by the body half-length l (unless otherwise stated). Associated with the centre-line the unit vectors in the tangential, normal, and binormal directions are given by (see figure 1)

$$\mathbf{e}_s = \frac{\partial \mathbf{x}_0}{\partial s}, \quad \mathbf{e}_n = \frac{1}{\kappa}\frac{\partial^2 \mathbf{x}_0}{\partial s^2}, \quad \mathbf{e}_b = \mathbf{e}_s \times \mathbf{e}_n, \tag{3}$$

where κ is the local non-dimensional curvature of the body centre-line. The admissible motions will be limited to those in which the centre-line radius of curvature is for all s and t large compared to the transverse cross-sectional radius of the body.

The motion of the body surface at any station s may be regarded as consisting of a translation of velocity $\mathbf{V}$ and a rotation about the point $\mathbf{x}_0(s,t)$ with angular velocity $\mathbf{\Omega}$. The no-slip boundary condition on the body surface at s is therefore given by

$$\mathbf{u} = \mathbf{V} + \mathbf{\Omega} \times r\mathbf{e}_r \quad \text{on} \quad r = \epsilon\eta(s), \tag{4}$$

where $\mathbf{e}_r = \sin\psi\,\mathbf{e}_b - \cos\psi\,\mathbf{e}_n$ and $r = \epsilon\eta(s)$ describes the body surface, with (r, ψ) denoting the polar co-ordinates in the body cross-sectional plane (figure 1). The velocity $\mathbf{V}$ and angular velocity $\mathbf{\Omega}$ are specified by

$$\mathbf{V}(s_0, t) = \left(\frac{d\mathbf{x}_0}{dt}\right)_{s_0\,\text{constant}} = \frac{\partial \mathbf{x}_0}{\partial t} + \frac{\partial \mathbf{x}_0}{\partial s}\frac{ds}{dt} = \frac{\partial \mathbf{x}_0}{\partial t} + \frac{ds}{dt}\mathbf{e}_s,$$

$$\left(\frac{d\mathbf{e}_\nu}{dt}\right)_{s_0\,\text{constant}} = \mathbf{\Omega} \times \mathbf{e}_\nu \quad (\nu = n, s, b),$$

where the velocity $\mathbf{V}$ and angular velocity $\mathbf{\Omega}$ have been non-dimensionalized by U and U/l respectively (U being an appropriate characteristic velocity). It is particularly convenient to decompose the boundary condition on $r = \epsilon\eta(s)$ into its components (u, v, w) along the $(\mathbf{e}_n, \mathbf{e}_s, \mathbf{e}_b)$ directions at s,

$$u = V_n - \epsilon\eta\Omega_s \sin\psi, \quad v = V_s + \epsilon\eta(\Omega_n \sin\psi + \Omega_b \cos\psi), \quad w = V_b - \epsilon\eta\Omega_s \cos\psi. \tag{5}$$

3. A slender prolate spheroid having a curved centre-line

(i) *Leading-order solution*

At present we restrict our attention to bodies with circular transverse cross-sections whose radius satisfies

$$r = \epsilon\eta(s) = \epsilon(1 - s^2)^{\frac{1}{2}} \quad (-1 \leqslant s \leqslant 1), \tag{6}$$

where as already mentioned we have non-dimensionalized with respect to l and defined $\epsilon = b/l \ll 1$. As a first approximation, we assume the velocity field is given by a line distribution of stokeslets and doublets along the body centre-line. The extent of the distribution is guided by the exact solution for a straight prolate spheroid (Chwang & Wu 1975) and taken to lie between the generalized foci of the body, i.e. the foci of the sketched straight body $s_1 = (1-\epsilon^2)^{\frac{1}{2}} = e$, where e is the generalized eccentricity. The velocity field is therefore given by

$$\mathbf{u}(s, r, \psi) = \int_{-s_1}^{s_1} (\mathbf{U}_S(\mathbf{R}; \boldsymbol{\alpha}) + \mathbf{U}_D(\mathbf{R}; \boldsymbol{\beta}))\, ds', \tag{7}$$

where s' is the integration variable and $\mathbf{U}_S$ and $\mathbf{U}_D$ are respectively the stokeslet and doublet velocity fields

$$\mathbf{U}_S = \frac{\boldsymbol{\alpha}}{R} + \frac{(\boldsymbol{\alpha}.\mathbf{R})\mathbf{R}}{R^3}, \quad \mathbf{U}_D = \frac{\boldsymbol{\beta}}{R^3} - \frac{3(\boldsymbol{\beta}.\mathbf{R})\mathbf{R}}{R^5}. \tag{8}$$

Here the stokeslet and doublet strengths per unit length, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, are unknown functions of s and t and are non-dimensionalized by U and Ul^2 respectively. Noting figure 1 we have

$$\left.\begin{aligned} \mathbf{R} &= \mathbf{R}_0 + r\mathbf{e}_r(s,t), \quad \mathbf{R}_0 = \mathbf{x}_0(s,t) - \mathbf{x}_0(s',t), \\ R &= |\mathbf{R}| = [R_0^2 + r^2 + 2r\mathbf{e}_r.\mathbf{R}_0]^{\frac{1}{2}}, \quad R_0 = |\mathbf{R}_0|, \end{aligned}\right\} \tag{9}$$

R being the position vector from the body centre-line at s' to a field point **x** in the plane of the polar co-ordinates (r, ψ) at s and $\mathbf{R}_0$ a position vector between the two points s' and s on the body centre-line.

In general, if we simply applied the boundary condition to (7) in its present form we would be left with the difficult job of solving a Fredholm integral equation of the first kind for the two singularity strengths. However, for the case of a slender body it is possible to partially integrate (7) by constructing uniformly valid expansions of the integrands for points in the neighbourhood of the body surface using the standard methods of perturbation theory (Fraenkel 1969; Cole 1968; Van Dyke 1975). This reduces (7) to the more easily solved second-kind integral equation for the stokeslet strength alone, with the doublet strength being determined in terms of the stokeslet strength. In particular, the uniformly valid expansion of the integrand is composed of the expansions in the following three regions: (1) the inner expansion for points s' near s where the boundary condition is to be satisfied, (2) the outer expansion for points s' away from s and (3) the common-part expansion which is the inner limit of the outer expansion or equivalently the outer limit of the inner expansion. The composite or uniformly valid expansion is then constructed by taking the sum of the inner and outer expansions and subtracting the common part. The resulting expansion can then be partially integrated termwise. This method is essentially a generalization of that used by Handelsman & Keller (1967), and Tillett (1970) for axisymmetric slender bodies. A complete account of the general expansion method is given by Johnson & Wu (1979). In the present study we will satisfy the boundary condition neglecting terms of $O(\epsilon^2)$ and therefore the integrands will be expanded to that order; the details of which are given in the appendix.

After integrating the expansions of the integrands in (7) (see appendix) and evaluating the velocity on the body surface neglecting terms of $O(\alpha\epsilon^2 \ln \epsilon)$ our first approximation to the velocity components in the $(\mathbf{e}_n, \mathbf{e}_s, \mathbf{e}_b)$ directions at s become

$$u \simeq 2(\alpha_n^{(0)} L + B_n^{(0)}) - f(s, \psi; \boldsymbol{\alpha}, \mathbf{B}) \cos\psi + u_n^{(0)} + \epsilon\eta(\cos\psi\, u^{(1)} + \sin\psi\, u^{(2)}), \qquad (10)$$

$$v \simeq 4(\alpha_s^{(0)} L - B_s^{(0)}) - g(s, \psi; \boldsymbol{\alpha}, \mathbf{B}) + u_s^{(0)} + \epsilon\eta(\cos\psi\, v^{(1)} + \sin\psi\, v^{(2)}), \qquad (11)$$

$$w \simeq 2(\alpha_b^{(0)} L + B_b^{(0)}) + f(s, \psi; \boldsymbol{\alpha}, \mathbf{B}) \sin\psi + u_b^{(0)} + \epsilon\eta(\cos\psi\, w^{(1)} + \sin\psi\, w^{(2)}), \qquad (12)$$

where $\alpha_\nu^{(0)}$ $(\nu = n, s, b)$ are the components of the first term in the inner expansion of the stokeslet strength (i.e. the stokeslet strength at $s' = s$), $B_\nu^{(0)}$ $(\nu = n, s, b)$ are the components of the first term in the inner expansion of the slowly varying portion of the doublet strength (see appendix, $\boldsymbol{\beta}(s) = \epsilon^2 \mathbf{B}(s)(s_1^2 - s^2)$), and

$$L = \ln \frac{2}{\epsilon},$$

$$f(s, \psi; \boldsymbol{\alpha}, \mathbf{B}) = h_1(s)\, G_s - h_2(s)\,(\cos\psi\, G_n - \sin\psi\, G_b),$$

$$g(s, \psi; \boldsymbol{\alpha}, \mathbf{B}) = h_2(s)\, G_s + h_1(s)\,(\cos\psi\, G_n - \sin\psi\, G_b),$$

$$G_\nu = \alpha_\nu^{(0)} - 2B_\nu^{(0)} \quad (\nu = n, s, b), \qquad (13)$$

$$h_1(s) = 2\epsilon\eta \frac{s}{1 - e^2 s^2}, \quad h_2(s) = 2\frac{1 - s^2}{1 - e^2 s^2}, \qquad (14)$$

$$u_\nu^{(0)} = \int_{-s_1}^{s_1} K_\nu(\mathbf{R}_0; \boldsymbol{\alpha})\, ds' \quad (\nu = n, s, b), \tag{15}$$

$$K_\nu(\mathbf{R}_0; \boldsymbol{\alpha}) = \frac{\alpha_\nu(s',t)}{R_0} + \frac{(\boldsymbol{\alpha}(s',t).\mathbf{R}_0)\, R_{0\nu}}{R_0^3} - \frac{D_\nu \alpha_\nu(s,t)}{|s-s'|} \tag{16}$$

$$(\nu = n, s, b; \quad \text{no sum on the subscript } \nu),$$

$$D_n = D_b = 1, \quad D_s = 2.$$

The vector components in the above integrand, K_ν, refer to the base vectors $\mathbf{e}_s$, $\mathbf{e}_n$, $\mathbf{e}_b$ at s where the boundary condition is to be satisfied. The velocity terms $u^{(k)}$, $v^{(k)}$ and $w^{(k)}$ $(k = 1, 2)$ in (10), (11) and (12) each have the general form

$$a(\boldsymbol{\alpha}) \ln \epsilon + b(\boldsymbol{\alpha}, \mathbf{B}) + c(\psi)\, G_\nu + \int_{-s_1}^{s_1} I(\mathbf{R}_0; \boldsymbol{\alpha})\, ds', \tag{17}$$

where G_ν is given by (13). The detailed expressions for (17) are extremely lengthy and of little practical interest and thus the reader is referred to Johnson (1977). The point to make here, for later use, is that all of the terms in (17) which are functions of ψ are found as products with the function $G_\nu = \alpha_\nu^{(0)} - 2B_\nu^{(0)}$, which will be equated to zero in satisfying the boundary conditions to leading order. Consequently only simple functions of ψ will remain in the terms of order ϵ in (10), (11) and (12).

We now observe that (10), (11) and (12) are capable of satisfying the boundary conditions (5) to the lowest order, with an error of $O(\alpha\epsilon \ln \epsilon)$, provided the angular velocity, $\boldsymbol{\Omega}$, is of $O(1)$ or smaller. With this restriction the terms involving $\boldsymbol{\Omega}$ in the boundary condition become of order ϵ or smaller. We then take $G_\nu = 0$, for all s, i.e.

$$B_\nu^{(0)} = \tfrac{1}{2}\alpha_\nu^{(0)} \quad (\nu = n, s, b), \tag{18}$$

in order to eliminate the leading-order terms in (10), (11) and (12) which depend on the angle ψ. We are then left with the following integral equation for $\boldsymbol{\alpha}$, neglecting terms of $O(\alpha\epsilon \ln \epsilon)$,

$$V_\nu(s,t) = \alpha_\nu(s,t)\, L_\nu + \int_{-s_1}^{s_1} K_\nu[\mathbf{R}_0; \boldsymbol{\alpha}(s',t)]\, ds' \quad (\nu = s, n, b), \tag{19}$$

where $L_s = 2(2L-1)$, $L_n = L_b = 2L+1$, $L = \ln(2/\epsilon)$ and $K_\nu(\mathbf{R}_0; \boldsymbol{\alpha})$ is given by (16). Consequently, we see that $\boldsymbol{\alpha} = O(1/\ln \epsilon)$. For a given centre-line configuration and motion the leading-order solution is fully determined when the solution of (19) is obtained. At present it appears that (19) determines $\boldsymbol{\alpha}$ to within an error of $O(\epsilon/\ln \epsilon)$; however, we will find, the rather surprising result that satisfying the boundary condition to higher orders in ϵ (neglecting $O(\epsilon^2 \ln \epsilon)$) does not alter this leading-order result for $\boldsymbol{\alpha}$ and consequently (19) determines $\boldsymbol{\alpha}$ to within an error of $O(\epsilon^2)$. A similar integral equation valid away from the body ends and in a somewhat different form is found to leading order using the method of matched asymptotic expansions by Keller & Rubinow (1976). Further discussion of this equation will be given later.

(ii) *Higher-order solution*

We now proceed to satisfy the boundary condition to higher orders in ϵ. We define the remaining velocity terms in the boundary condition after neglecting the terms of $O(\epsilon^2)$ as

$$\left.\begin{aligned}\hat{u} &= \epsilon\eta[\cos\psi\, u^{(1)} + \sin\psi(u^{(2)} + \Omega_s)],\\ \hat{v} &= \epsilon\eta[\cos\psi(v^{(1)} - \Omega_b) + \sin\psi(v^{(2)} - \Omega_n)],\\ \hat{w} &= \epsilon\eta[\cos\psi(w^{(1)} + \Omega_s) + \sin\Omega\, w^{(2)}].\end{aligned}\right\} \quad (20)$$

In order to satisfy the boundary condition to higher orders it will be necessary to cancel the above residual velocities on the body surface. This will be done by incorporating further singularity distributions into our solution.

We begin by considering the velocity components in the n–b or transverse cross-sectional plane of the body, i.e. $\hat{u}$ and $\hat{w}$. These components can be written as

$$\left.\begin{aligned}\hat{u} &= \epsilon\eta[(-a+b)\cos\psi + (-c+d)\sin\psi],\\ \hat{w} &= \epsilon\eta[(a+b)\sin\psi + (c+d)\cos\psi],\end{aligned}\right\} \quad (21)$$

where

$$\left.\begin{aligned}a &= \tfrac{1}{2}(w^{(2)} - u^{(1)}), \quad b = \tfrac{1}{2}(w^{(2)} + u^{(1)}),\\ c &= \tfrac{1}{2}(w^{(1)} - u^{(2)}), \quad d = \tfrac{1}{2}(w^{(1)} + u^{(2)} + 2\Omega_s).\end{aligned}\right\} \quad (22)$$

The terms are easily identified by combining the velocities $\hat{u}$ and $\hat{w}$ and expressing the result in terms of the $(\mathbf{e}_r, \mathbf{e}_\psi)$ base vectors (see figure 1),

$$\hat{u}\mathbf{e}_n + \hat{w}\mathbf{e}_b = a\mathbf{e}_r - b(\cos 2\psi\,\mathbf{e}_r - \sin 2\psi\,\mathbf{e}_\psi) + c(\sin 2\psi\,\mathbf{e}_r + \cos 2\psi\,\mathbf{e}_\psi) + d\mathbf{e}_\psi. \quad (23)$$

A physical interpretation of the velocity terms in (23) clearly motivates the choice of additional singularities to be used. The first term with coefficient a is a 'radial-type' flow in the transverse cross-sectional plane which exhibits a constant flow in the $\mathbf{e}_r$ direction for all angles ψ. Such a velocity term obviously suggests that we incorporate into our solution a source distribution or equivalently a tangential doublet distribution. The last term in (23) is a 'rotational-type' flow about the body centre-line and can be corrected for by including a tangential rotlet distribution with strength $\hat{\delta}_s$. The remaining two terms represent 'extensional-type' flows in the cross-sectional plane, with their principal axes subtending an angle of $\frac{1}{4}\pi$. Based on the exact solution for a two-dimensional circular cylinder in an extensional flow (Chwang & Wu (1975) we envisage the need for a stresslet and quadrupole distribution to eliminate each of these terms.

The higher-order tangential velocity term, $\hat{v}$, represents two 'shear-like' flows in the $\mathbf{e}_s$ direction and can be adjusted to satisfy the boundary condition by introducing a distribution of rotlets oriented in the $\mathbf{e}_n$ and $\mathbf{e}_b$ directions (strengths $\hat{\delta}_n$, $\hat{\delta}_b$).

It is worthwhile to note that if we only desire to satisfy the boundary conditions through terms of $O(\alpha\epsilon\ln\epsilon)$, with an error of $O(\alpha\epsilon)$, then the explicit expressions for $u^{(k)}$, $v^{(k)}$ and $w^{(k)}$ (17) are given by (Johnson 1977)

$$\left.\begin{aligned}v^{(1)} &= 2\left(\frac{\partial\alpha_n}{\partial s} - \alpha_s\kappa + \tau\alpha_b\right)\ln\frac{2}{\epsilon},\\ v^{(2)} &= -2\left(\frac{\partial\alpha_b}{\partial s} - \tau\alpha_n\right)\ln\frac{2}{\epsilon},\\ u^{(1)} &= -w^{(2)} = \left(2\frac{\partial\alpha_s}{\partial s} - \kappa\alpha_n\right)\ln\frac{2}{\epsilon},\\ u^{(2)} &= w^{(1)} = -\alpha_b\kappa\ln\frac{2}{\epsilon},\end{aligned}\right\} \quad (24)$$

where τ is the centre-line torsion (see appendix). Consequently we then have $b = c = 0$ in (22) and therefore to that order only a rotlet and source distribution is needed since the 'extensional-type' velocity terms are of higher order. In addition, at this order we have the first appearance of the local centre-line curvature effect.

The velocity field induced by the higher-order singularities used to correct the boundary condition is

$$\mathbf{u}^* = \int_{-s_1}^{s_1} \{\mathbf{U}_R(\mathbf{R}; \hat{\boldsymbol{\delta}}) + \mathbf{U}_{\text{source}}(\mathbf{R}; \hat{m}) + \hat{A}_1 \mathbf{U}_{SS}(\mathbf{R}; \mathbf{e}_n, \mathbf{e}_n) + \hat{B}_1 \mathbf{U}_Q(\mathbf{R}; \mathbf{e}_n, \mathbf{e}_n) + \hat{A}_2 \mathbf{U}_{SS}(\mathbf{R}; \mathbf{e}_n, \mathbf{e}_b) + \hat{B}_2 \mathbf{U}_Q(\mathbf{R}; \mathbf{e}_n, \mathbf{e}_b)\}\, ds'. \quad (25)$$

The general form of the rotlet, source, stresslet and quadrupole velocity terms are given respectively by (Chwang & Wu 1975)

$$\left.\begin{aligned} \mathbf{U}_R(\mathbf{R}; \hat{\boldsymbol{\delta}}) &= \frac{\hat{\boldsymbol{\delta}} \times \mathbf{R}}{R^3}, \quad \mathbf{U}_{\text{source}}(\mathbf{R}; \hat{m}) = \frac{\hat{m}\mathbf{R}}{R^3}, \\ \mathbf{U}_{SS}(\mathbf{R}; \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) &= \left\{-\frac{\boldsymbol{\mu}_1 . \boldsymbol{\mu}_2}{R^3} + \frac{3(\boldsymbol{\mu}_1 . \mathbf{R})(\boldsymbol{\mu}_2 . \mathbf{R})}{R^5}\right\}\mathbf{R}, \\ \mathbf{U}_Q(\mathbf{R}; \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) &= (\boldsymbol{\mu}_1 . \nabla)(\boldsymbol{\mu}_2 . \nabla)\, \nabla\left(\frac{1}{R}\right). \end{aligned}\right\} \quad (26)$$

For the special case of a torus it was found (Johnson & Wu 1979) that the rotlet, source and stresslet strengths were each proportional to the cross-sectional radius squared, while the quadrupole strength was proportional to the cross-sectional radius raised to the fourth power. With this guidance we assume that the rotlet, source, stresslet, and quadrupole strengths are respectively given by

$$\left.\begin{aligned} \hat{\boldsymbol{\delta}}(s,t) &= \epsilon^2 \boldsymbol{\delta}(s,t)\,(s_1^2 - s^2), \\ \hat{m}(s,t) &= \epsilon^2 m(s,t)\,(s_1^2 - s^2), \\ \left.\begin{aligned} \hat{A}_k(s,t) &= \epsilon^2 A_k(s,t)\,(s_1^2 - s^2) \\ \hat{B}_k(s,t) &= \epsilon^4 B_k(s,t)\,(s_1^2 - s^2)^2 \end{aligned}\right\} &\quad \text{for} \quad k = 1, 2. \end{aligned}\right\} \quad (27)$$

As one might expect from the weak long-range behaviour of these singularities the outer and common part expansion of the above singularities are all of higher order than those terms retained. In particular it is easy to verify after determining the singularity strengths that the outer and common-part expansions yield a contribution to the velocity field of $O(\epsilon^2)$ or smaller. We construct the inner expansions to the integrand by expanding the singularity strengths in the two-variable fashion as was done for the doublet strength (see appendix). After integrating and evaluating on the body surface we find, after neglecting terms of $O(\epsilon^2)$,

$$\left.\begin{aligned} u^* &\simeq -2\epsilon\eta[(D_2 - C_2)\sin\psi + (2B_1 + D_1)\cos\psi] - 2\epsilon^2\eta^2 \ln\frac{2}{\epsilon}\left[\frac{2}{\eta}\frac{\partial\eta}{\partial s}\delta_b + \frac{\partial\delta_b}{\partial s} - \tau\delta_n + \frac{\kappa}{2}m\right], \\ v^* &\simeq 2\epsilon\eta(\delta_b\cos\psi + \delta_n\sin\psi) - 4\epsilon^2\eta^2 \ln\frac{2}{\epsilon}\left[\frac{1}{\eta}\frac{\partial\eta}{\partial s}m + \frac{1}{2}\left(\frac{\partial m}{\partial s} - \frac{\kappa}{2}\delta_b\right)\right], \\ w^* &\simeq -2\epsilon\eta[(D_2 + C_2)\cos\psi + (2B_1 - D_1)\sin\psi] + 2\epsilon^2\eta^2 \ln\frac{2}{\epsilon}\left[\frac{2}{\eta}\frac{\partial\eta}{\partial s}\delta_n + \frac{\partial\delta_n}{\partial s} + \tau\delta_b + \frac{\kappa}{2}\delta_s\right], \end{aligned}\right\} \quad (28)$$

where

$$C_k = A_k - 2B_k \quad (k = 1, 2),$$
$$D_1 = m + (A_1 - 4B_1)\cos 2\psi,$$
$$D_2 = \delta_s - (A_2 - 4B_2)\cos 2\psi,$$

and $\eta^2 = 1 - s^2$ on the body surface. We now observe that the first terms in the above expressions (those terms in (28) containing the factor $2\epsilon\eta$) can correct for the residual velocity terms in (20) and thereby satisfy the boundary condition through terms of $O(\alpha\epsilon)$ by requiring

$$\left.\begin{aligned} &\delta_s = \tfrac{1}{2}d = \tfrac{1}{4}(w^{(1)} + u^{(2)} + 2\Omega_s), \quad \delta_b = \tfrac{1}{2}(\Omega_b - v^{(1)}), \quad \delta_n = \tfrac{1}{2}(\Omega_n - v^{(2)}),\\ &m = -\tfrac{1}{2}a = \tfrac{1}{4}(u^{(1)} - w^{(2)}),\\ &A_1 = b = \tfrac{1}{2}(w^{(2)} + u^{(1)}), \quad B_1 = \tfrac{1}{4}A_1,\\ &A_2 = c = \tfrac{1}{2}(w^{(1)} - u^{(2)}), \quad B_2 = \tfrac{1}{4}A_2.\end{aligned}\right\} \tag{29}$$

As already mentioned, up to this point we have been able to satisfy the no-slip boundary condition neglecting terms of $O(\epsilon^2 \ln \epsilon)$ without altering the stokeslet strength, $\boldsymbol{\alpha}$, determined by the integral equation found earlier. Therefore the integral equation (19) actually determines $\boldsymbol{\alpha}$ within an error of $O(\epsilon^2)$. However, in order to correct for the additional velocity terms induced by the rotlet and source distribution, i.e. the remaining terms of $O(\epsilon^2 \ln \epsilon)$ in (28), it is necessary to make a correction of $O(\epsilon^2)$ to the stokeslet strength. We therefore include an additional stokeslet with strength $\epsilon^2\boldsymbol{\alpha}^*$ in our solution. Clearly from (10), (11) and (12) (see appendix) the leading-order velocity term due to the new stokeslet will be

$$u \simeq 2\epsilon^2 \ln\frac{2}{\epsilon}\,\alpha_n^*, \quad v \simeq 4\epsilon^2 \ln\frac{2}{\epsilon}\,\alpha_s^*, \quad w \simeq 2\epsilon^2 \ln\frac{2}{\epsilon}\,\alpha_b^*. \tag{30}$$

Thus the remaining terms in (28) are cancelled by taking

$$\left.\begin{aligned} \alpha_n^* &= 2\eta\frac{\partial\eta}{\partial s}\,\delta_b + \eta^2\left(\frac{\partial\delta_b}{\partial s} - \tau\delta + \frac{\kappa}{2}\,m\right),\\ \alpha_s^* &= \eta\frac{\partial\eta}{\partial s}\,m + \tfrac{1}{2}\eta^2\left(\frac{\partial m}{\partial s} - \frac{\kappa}{2}\,\delta_b\right),\\ \alpha_b^* &= -2\eta\frac{\partial\eta}{\partial s}\,\delta_n - \eta^2\left(\frac{\partial\delta_n}{\partial s} + \tau\delta_b + \frac{\kappa}{2}\,\delta_s\right).\end{aligned}\right\} \tag{31}$$

4. Slender bodies having arbitrary longitudinal cross-sections away from the ends

In a straightforward manner we can modify the previous solution method to handle slender bodies whose longitudinal cross-section is arbitrary over the majority of the body length and approximately elliptic in the vicinity of the body ends. In particular the cross-section is specified by

$$r = \epsilon\eta(s),$$

with the condition that $\eta^2(s) \sim (1 - s^2)(1 + O(\epsilon^2))$ as $s \to \pm 1$ and $\eta(s)$ is a sufficiently slowly varying function of s away from the ends. The primary difference in this case is that we now construct expansions to the velocity field near the body surface in two regions, one valid in the centre of the body away from the ends and the other for the end regions where $s \sim \pm 1$. Ultimately we then match the centre and end expansions in their common region and construct the uniformly valid expansion for the velocity field in the standard fashion. For each of the regions, i.e. centre and ends, the integrands of the velocity field singularity distributions are handled precisely as in §3; namely we construct uniformly valid expansions about the point $s' = s$. Since the methods follow from before we will, for the sake of brevity, simply point out the modifications to be made.

The velocity field is constructed using the same singularities as before, except here we assume that all of the singularity strengths, excluding the stokeslet, are specified in the two regions as

$$f(s,t) = \begin{cases} \epsilon^n f(s,t)\,\eta^n(s) & \text{(centre region)}, \\ \epsilon^n f(s,t)\,(s_1^2 - s^2)^{n/2} & \text{(end region)}, \end{cases}$$

where $f(s,t)$ represents the various singularity strengths and $n = 2$ for all of the higher-order singularities except for the quadrupole, in which case $n = 4$. The stokeslet strength is expressed in precisely the same manner as before over the entire body length.

In the end regions the inner expansion of the integrands is constructed by applying the two-variable expansion technique to the singularity strengths as was done in §3. In these regions the inner and outer expansions of the integrands are identical with those in §3 (also see appendix) since the expansions of the strengths and cross-sectional shapes are the same. Consequently the velocity field in the neighbourhood of the body surface near the ends is unchanged from before (equations (10), (11), (12)). The boundary condition in the end regions is therefore satisfied by (18) and (19) and is extended to higher orders in ϵ giving the same results as in §3.

For the centre region a two-variable expansion of the singularity strengths is unnecessary since the cross-sectional radius is slowly varying and therefore we use a Taylor series expansion about $s' = s$. Here the inner expansion of the integrands, excluding the stokeslet, take a new form since the singularity strengths are no longer expanded in a two-variable fashion. As a result of this and the fact that in the centre region the cross-sectional shape is given by the general expression $\epsilon\eta(s)$ we find after integration and evaluation on the body surface that the velocity due to the stokeslet and doublet is given by (10), (11) and (12) with the following changes:

$$\left.\begin{aligned} & h_2(s) \to 2, \quad L \to \ln \frac{2(1-s^2)^{\frac{1}{2}}}{\epsilon\eta(s)}, \\ & h_1(s)\,G_\nu \to 2\epsilon\eta\left[\frac{s}{1-s^2}\,\alpha_\nu^{(0)} + 2B_\nu^{(0)}\,\frac{1}{\eta}\frac{\partial\eta}{\partial s}\right]. \end{aligned}\right\} \tag{32}$$

In the centre region the no-slip boundary condition is then satisfied at the leading order (neglecting terms of $O(\alpha\epsilon\ln\epsilon)$) by taking $G_n = G_b = 0$ and the solution of $\boldsymbol{\alpha}$ from the integral equation (19) where L assumes the new form given in (32). Furthermore the matching condition on the velocity field, i.e. the end limit of the centre expansion must equal the centre limit of the end expansion, requires $G_s = 0$. For the centre region the higher-order velocity terms $u^{(k)}$, $v^{(k)}$ and $w^{(k)}$ [equation (17)] resulting from the stokeslet and doublet distribution now contain an additional term of the form

$$F(\boldsymbol{\alpha})\left[\frac{s}{1-s^2}+\frac{1}{\eta}\frac{\partial \eta}{\partial s}\right]$$

which clearly vanishes for the special case considered in § 3. With only this slight change occurring, the boundary condition in the centre region can also be satisfied through terms of $O(\epsilon^2)$ by including the higher-order singularities into the solution. The algebraic details are given by Johnson (1977).

After constructing the uniformly valid expansions to the velocity field on the body surface you find that the integral equation for $\boldsymbol{\alpha}$ [equation (19)] with L given by (32) is valid along the entire body length, determining $\boldsymbol{\alpha}$ with an error of $O(\epsilon^2)$. Note that near the ends the behaviour of $\eta(s)$ is such that $\ln[2(1-s^2)^{\frac{1}{2}}/\epsilon\eta(s)] \sim \ln(2/\epsilon)$ and we obtain the integral equation for an elliptical cross-section body. The order ϵ^2 correction to the stokeslet strength is given as before by (31), except here the body cross-sectional shape $\eta^2(s) \neq 1-s^2$.

5. Applications

In many applications concerning the motion of slender bodies at low Reynolds numbers the quantity of interest is the total hydrodynamic force $\mathbf{F}$ and the force per unit length $\mathbf{f} = \mathbf{F}/ds$ acting on the body. The first of these follows from the well-known result that the total force on the fluid is $8\pi\mu$ times the total stokeslet strength. It does not, however, immediately follow that the force per unit length along the entire body length is given by $8\pi\mu$ times the stokeslet strength per unit length, as $\boldsymbol{\alpha}$ is not distributed along the entire body length. However, from the expression for the force exerted on a closed surface in an incompressible Newtonian fluid,

$$\mathbf{F} = \int_{S_b} (-p\mathbf{n}+\mu\boldsymbol{\omega}\times\mathbf{n})\, dS,$$

where p is the pressure, $\boldsymbol{\omega}$ the vorticity and $\mathbf{n}$ the outward unit normal to the body surface S_b, it can be shown after a rather lengthy calculation (making use of the expressions for the p and $\boldsymbol{\omega}$ due to the various singularity distributions) that the force per unit length is given with an error of $O(\epsilon^2)$ by $8\pi\mu\boldsymbol{\alpha}$ (Johnson 1977). It is worthwhile to point out that this result is not necessarily true to higher orders in the slenderness parameter. This can be seen from the exact solution for a straight centre-line prolate spheroid (Chwang & Wu 1975) which gives $\mathbf{f} = 8\pi\mu\boldsymbol{\alpha}(1-\epsilon^2)^{\frac{1}{2}}$. For practical application an error in the force per unit length of $O(\epsilon^2)$ is often sufficient and therefore the present interest is in discussing methods available for calculating the stokeslet strength from the integral equation (19), repeated here for convenience,

$$V_\nu(s,t) = \alpha_\nu(s,t)\, L_\nu + \int_{-s_1}^{s_1} K_\nu(\mathbf{R}_0;\boldsymbol{\alpha})\, ds' \quad (\nu = s,n,b), \tag{33}$$

where

$$K_\nu(\mathbf{R}_0;\boldsymbol{\alpha}) = \frac{\alpha_\nu(s',t)}{R_0}+\frac{(\boldsymbol{\alpha}(s',t).\mathbf{R}_0)\, R_{0\nu}}{R_0^3}-\frac{D_\nu\, \alpha_\nu(s,t)}{|s-s'|},$$

$L_s = 2(2L-1)$, $L_n = L_b = 2L+1$, $L = \ln[2(1-s^2)^{\frac{1}{2}}/\epsilon\eta]$, $D_n = D_b = 1$, $D_s = 2$, and $\mathbf{R}_0$ is the vector between the points s' and s on the body centre-line.

Clearly (33) is in general three simultaneous integral equations for the three vector components of the stokeslet strength, $\alpha_n, \alpha_s, \alpha_b$. However, for bodies with a planar centre-line, i.e. bodies for which $\mathbf{e}_n$ and $\mathbf{e}_s$ lie in the same plane for all s, we see that $R_{0b} \equiv 0$ and thus the equation for α_b separates from the equations for α_s and α_n. Consequently if $V_b = 0$ we have the expected trivial solution $\alpha_b = 0$ and there remains two coupled equations for α_n and α_s. Similarly if $V_n = V_s = 0$, but $V_b \neq 0$ we have the solution $\alpha_n = \alpha_s = 0$ and a single integral equation for α_b. For a straight centre-line body (33) reduces to three independent equations for $\alpha_n, \alpha_s, \alpha_b$ which has the well-known solution for a slender prolate spheroid $\alpha_\nu = V_\nu/L_\nu$. One further comment concerning (33) is that for a toroidal ring of constant cross-section equation (33) is valid if the integration variable is taken to be the angle ϕ between s and s' with the integration limits $\pm\pi$ and $L = \ln(2/\epsilon)$. In that particular case Johnson & Wu (1979) have, by a somewhat different approach, solved the integral equation in closed form. However, for more general body shapes having ends it is usually necessary to resort to approximate numerical techniques.

One rather obvious approximate procedure, that can be carried out analytically in some cases, is to use the iteration scheme defined by

$$\alpha_\nu^{(k+1)}(s,t) = \frac{1}{L_\nu}\left\{V_\nu(s,t) - D_\nu \ln f(s)\, \alpha_\nu^{(k)}(s,t) - \int_{-s_1}^{s_1} K_\nu(\mathbf{R}_0; \boldsymbol{\alpha}^{(k)})\, ds'\right\}, \tag{34}$$

where $\alpha_\nu^{(k)}$ $(\nu = n,s,b)$ is the kth iteration of α_ν, $f(s) = (1-s^2)/\eta^2(s)$, and here $L = \ln(2/\epsilon)$ in L_ν. With the initial guess $\alpha_\nu^{(0)} = 0$ we see that $\alpha_\nu^{(1)}$ gives the familiar result for a straight, slender prolate spheroid. Continuing the iteration to higher orders yields succeeding corrections to the solution sought for the actual body shape and motion. This iteration scheme generates an expansion for α_ν in powers of $1/L_\nu$ with an error in the kth iteration of order $(1/L_\nu)^{k+1}$. The expansion here arises as the natural one for the problem and appears to be of a somewhat more general nature than the expansions in terms of $(1/\ln\epsilon)^n$ given by Cox (1970) and Keller & Rubinow (1976). In fact, if we have the same number of terms in the two expansions we must neglect an infinite number of terms in the $1/L_\nu$ expansion to obtain the expansion in terms of $1/\ln\epsilon$.

The iteration method given by (24) has an error of $O(1/L_\nu)^{k+1}$ in the kth iteration and therefore the sequence is slowly convergent. Accordingly, many iterations may often be needed in practical application to obtain sufficiently accurate results. Also we see that it is impossible with this method to take full advantage of the true accuracy of $\boldsymbol{\alpha}$ inherent in the original integral equation (i.e. $O(\epsilon^2)$). For this reason it is often desirable to use a direct numerical computation method. The procedure used in the examples to follow was to generate a set of linear equations by replacing the integral in (33) with a sum using a quadratic quadrature formula. The linear system of equations was then easily solved using either Gaussian elimination with iterative improvement or the Gauss–Seidel iteration method. We note that for the case of a slender torus and straight prolate spheroid the results from this computational method were in excellent agreement with the analytic results.

A useful example demonstrating the iteration procedure given by (34) is the translation of a toroidal ring along its symmetry axis. For this case $V_n = V_s = 0$, $V_b = 1$, and therefore $\alpha_n = \alpha_s = 0$. With the appropriate changes to the integral equation for a torus we have, after taking $\alpha_b^{(0)} = 0$,

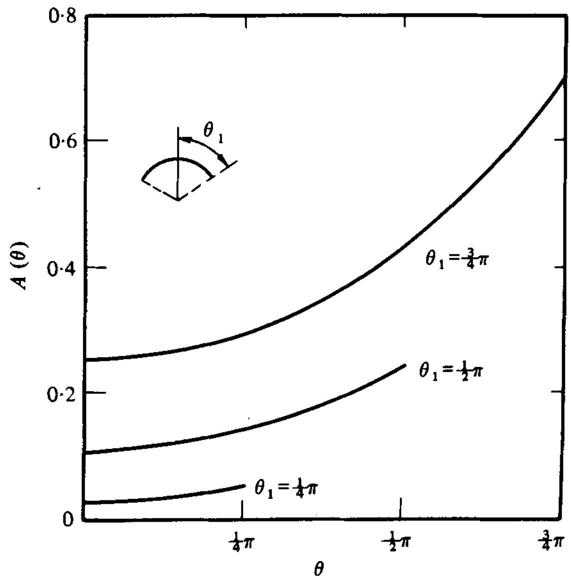

FIGURE 2. Second iteration for the broadwise motion of a partial torus.

$$\alpha_b^{(2)} = \frac{1}{L_b} - \frac{1}{L_b^2}\int_{-\pi}^{\pi}\left\{\frac{1}{|2\sin\frac{1}{2}\phi|} - \frac{1}{|\phi|}\right\}d\phi = \frac{1}{L_b} + \frac{2\ln\frac{1}{4}\pi}{L_b^2}. \tag{35}$$

It is easy to see that continuing the iteration to higher orders will lead to similar terms, thus giving

$$\alpha_b^{(k)} = \frac{1}{L_b}\sum_{n=0}^{k-1}\left(\frac{2\ln\frac{1}{4}\pi}{L_b}\right)^n = \frac{1}{L_b - 2\ln\frac{1}{4}\pi}.$$

The final expression above is the summed result for an infinite number of iterations and is precisely the result found in a direct fashion by Johnson & Wu (1979).

We can also easily obtain the first two terms for the similar translation of a partial spheroidal torus, i.e. a body with an elliptical longitudinal cross-section and a centre-line given in polar co-ordinates by $\rho = a$, $|\theta| \leqslant \theta_0$ (i.e. an arc of a circle). For this case the second iteration is also given by (35) although now the integration limits are given by the body ends, $-e\theta_0 - \theta$ and $e\theta_0 - \theta$, where e is the generalized eccentricity. Integrating, we find

$$\alpha_b^{(2)} = \frac{1}{L_b}\left(1 - \frac{A(\theta)}{L_b}\right) + O\left(\frac{1}{L_b}\right)^3. \tag{36a}$$

where

$$A(\theta) = \ln\frac{16}{\theta_1^2 - \theta^2}\tan\frac{\theta_1 + \theta}{4}\tan\frac{\theta_1 - \theta}{4} \tag{36b}$$

and $\theta_1 = e\theta_0$. From figure 2, where we have plotted A for some typical cases, we observe that this first correction results in decreasing the force per unit length from that on a straight centre-line body (i.e. $1/L_b$) with the greatest reduction near the body ends. As expected, for small θ_1, $A(\theta)$ is small and the result is close to that for a straight body. We note that the difference between (36) and the solution obtained by the direct numerical method (not shown) is found to be less than 1%. In this simple case the

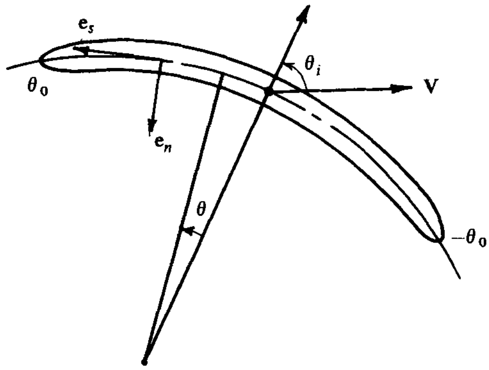

FIGURE 3. Translation of a partial torus in its own plane.

accuracy of the iteration scheme is quite surprising, however, for more complex body centre-lines and motions it is overly optimistic to expect such a high degree of accuracy with so few iterations.

An example, which can be compared to a result given by Cox (1970), is the translation of a partial spheroidal torus in its own plane. With the translational velocity taken to be $\mathbf{V} = \mathbf{e}_x$ we have

$$V_n = -\cos\psi, \quad V_s = -\sin\psi, \quad V_b = 0,$$

where $\psi = \theta+\theta_i$, $\theta = 0$ being the centre-line midpoint and θ_i the angle of incidence between $\theta = 0$ and the direction of translation $\mathbf{e}_x$ (figure 3). Here we have $\alpha_b = 0$ and with the initial guess $\boldsymbol{\alpha}^{(0)} = 0$ we find

$$\alpha_\nu^{(2)}(\theta) = \frac{V_\nu}{L_\nu} - \frac{1}{L_\nu^2}(V_n g_\nu(\theta) + V_s k_\nu(\theta)) \quad (\nu = n, s), \tag{37}$$

where

$$g_n(\theta) = A + a(1-3\mathscr{L}) + c(1-\mathscr{L}) - 2(1-2\mathscr{L}),$$

$$k_n(\theta) = -b(1-3\mathscr{L}) - d(1-\mathscr{L}),$$

$$g_s(\theta) = b(5-\mathscr{L}^{-1}) + d(1-\mathscr{L}^{-1}),$$

$$k_s(\theta) = 2A + a(5-3\mathscr{L}^{-1}) + c(1-\mathscr{L}^{-1}) - 2(3-2\mathscr{L}^{-1}),$$

$$a = \cos\tfrac{1}{2}\theta_1 \cos\tfrac{1}{2}\theta, \quad b = -\cos\tfrac{1}{2}\theta_1 \sin\tfrac{1}{2}\theta,$$

$$c = \cos\tfrac{3}{2}\theta_1 \cos\tfrac{3}{2}\theta, \quad d = -\cos\tfrac{3}{2}\theta_1 \sin\tfrac{3}{2}\theta,$$

$\mathscr{L} = L_n/L_s$ and A is given in (36b). Cox (1970) gives the x component of the force per unit length for this case [his equation (7.32)] as an expansion in terms of $1/\ln\epsilon$. Here the non-dimensional force per unit length on the fluid in the x direction is given by $f_x = -8\pi(\alpha_n\cos\psi + \alpha_s\sin\psi)$, where α_n and α_s are given by (37). Cox's result can be obtained from the present result by expanding $1/L_\nu$ $(\nu = n, s)$ in terms of $1/\ln\epsilon$, neglecting $O[(1/\ln\epsilon)^3]$, and by approximating the result for points away from the body ends. In figure 4 Cox's result is compared with the second iteration given here and to the direct numerical computation of $\boldsymbol{\alpha}$ for a typical case. We see that the expansion in $1/L_\nu$ is generally in better agreement with the numerical result than the expansion

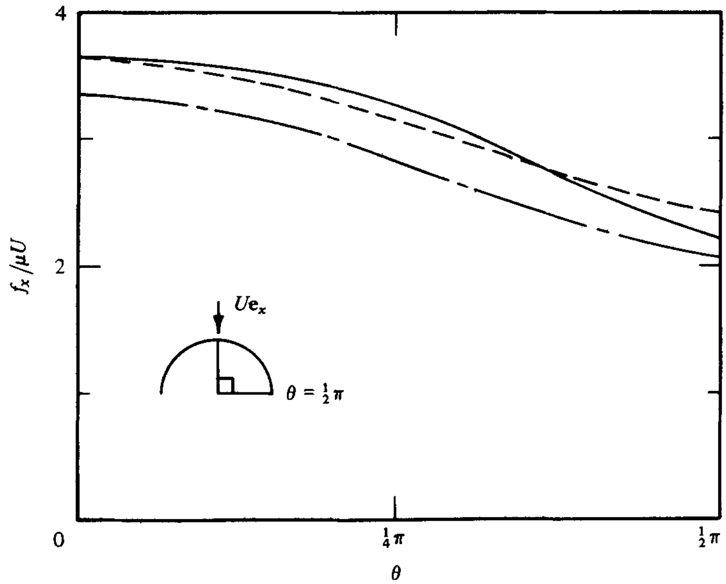

FIGURE 4. Drag force per unit length for a partial torus translating as shown ($\epsilon = 0{\cdot}1$). ——, numerical solution; – – –, $1/L_\nu$ expansion; — — —, $1/\ln\epsilon$ expansion.

given by Cox. Apparently the fact that the terms $1/L_\nu$ sum an infinite number of terms absent in the $1/\ln\epsilon$ expansion make it a better approximation. Near the body ends both expansions have approximately the same error magnitude; therefore it would seem that when end effects are of primary interest it is generally best to abandon the iteration scheme and resort to the direct numerical approach.

An example relevant to flagellar hydrodynamics is the motion of a slender body having a helical centre-line. Of particular interest is the translation of the helix along its axis. Here we will compare the results of the slender-body theory (using the direct numerical calculation method) to those of resistive force theory using (1) the classical Gray & Hancock force coefficients and (2) the zero thrust sub-optimal force coefficients recently introduced by Lighthill (1976). From the resistive-force theory the force per unit length, $\mathbf{f}$, exerted on the fluid by a slender body with centre-line velocity $\mathbf{V}$ is $f_\nu = \mu C_\nu V_\nu$ ($\nu = n, s, b$), where C_ν is the force coefficient. The coefficients of Gray & Hancock and Lighthill are respectively given by

$$C_s = \frac{2\pi}{\ln\dfrac{2\lambda}{b}-\dfrac{1}{2}}, \quad C_b = \frac{4\pi}{\ln\dfrac{2\lambda}{b}+\dfrac{1}{2}} \quad \text{(Gray \& Hancock)},$$

$$C_s = \frac{2\pi}{\ln\dfrac{2q}{b}}, \quad C_b = \frac{4\pi}{\ln\dfrac{2q}{b}+\dfrac{1}{2}} \quad \text{(Lighthill)},$$

where $q = 0{\cdot}09\Lambda$, Λ being the wavelength measured along the body centre-line, b the transverse cross-sectional radius and λ the wavelength measured along the helix axis. The need for a coefficient C_n does not arise since $V_n = 0$ for the translation along the helix axis (the direction n being radially inward to the helix axis or z axis in a cylindrical co-ordinate system (ρ, θ, z)).

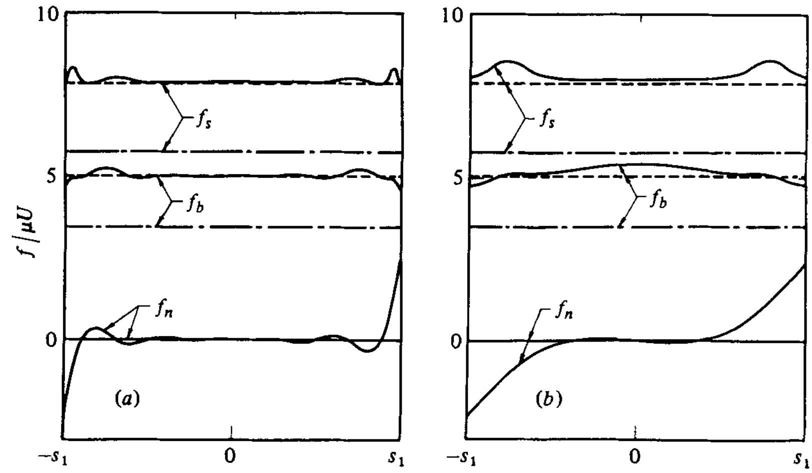

FIGURE 5. The force per unit length for the translation of a helix along its symmetry axis at the zero thrust values of c/U (constant cross-section). (a) Five waves along the body length, $c/U = 2{\cdot}90$; (b) one wave, $c/U = 2{\cdot}92$. ——, slender-body theory; ---, Lighthill (1976); —·—, Gray & Hancock. Wave parameters: $a/\lambda = 0{\cdot}25$, $b/\lambda = 0{\cdot}01$.

The Lighthill force coefficients are valid for a helical flagellum producing zero net thrust, i.e. a flagellum with negligible cell body drag to overcome. Therefore an appropriate comparison is made by considering a helix translating along its symmetry axis with its angular velocity, ω, about that axis chosen such that the total force in the translational direction is zero. For a detailed discussion concerning the propulsion of flagellated micro-organisms possessing helical flagellal the reader is referred to Lighthill (1976), Chwang & Wu (1971), Chwang, Winet & Wu (1974) and Higdon (1979). The helix centre-line and motion under consideration is described in an orthogonal Cartesian co-ordinate system $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ by

$$\mathbf{x} = (a\sin\theta, a\cos\theta, \theta/k), \quad \theta/k = \frac{s+l}{(1+a^2k^2)^{\frac{1}{2}}},$$

$$\mathbf{V} = (a\omega\cos\theta, \, -a\omega\sin\theta, U),$$

where a is the amplitude, $k = 2\pi/\lambda$, U the translational or swimming speed and the wave speed c is given by ω/k.

In figures 5(a, b) we compare the force per unit length exerted on the fluid calculated by the slender-body theory and the resistive-force theory for a representative case. The body cross-section was constant over the major portion of the centre-line and fitted to a prolate spheroid at a distance 20ϵ from the body ends. There are five waves (or turns) along the helix in figure 5(a) ($n = 5$) and one wave in figure 5(b) ($n = 1$). As already noted the value of ω or equivalently c/U, which is indicated in each figure, was determined from the slender-body theory in order to obtain zero thrust. Furthermore, the zero thrust value of c/U calculated with Lighthill's coefficients was within 2 % of the value shown, whereas the Gray & Hancock coefficients were in very poor agreement.

Although Lighthill's coefficients were derived for a helix with an infinite number of waves along its length (i.e. neglecting end effects) we see in figure 5(a) that excellent

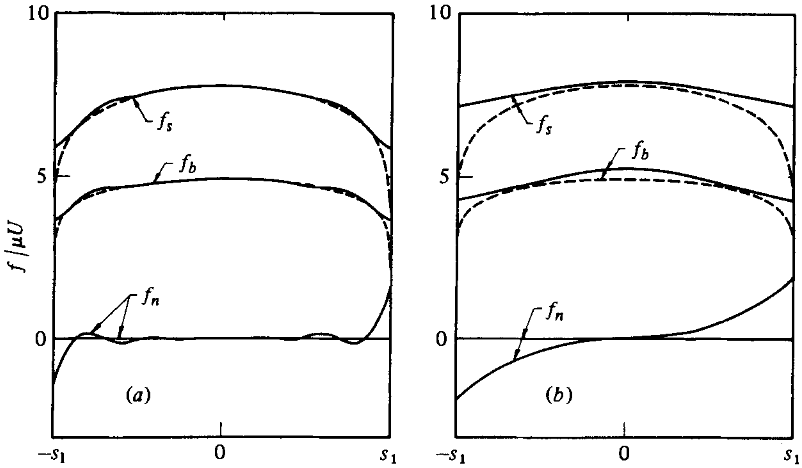

FIGURE 6. The force per unit length for the translation of a helix along its symmetry axis at the zero thrust value of c/U (prolate spheroidal cross-section). (*a*) Five waves along the body length, $c/U = 2{\cdot}86$; (*b*) one wave, $c/U = 2{\cdot}87$. ——, slender-body theory; ---, Lighthill (1976). Wave parameters: $a/\lambda = 0{\cdot}25$, $b/\lambda = 0{\cdot}01$.

agreement with the slender-body theory is obtained for a helix with only five waves. As expected, the only deviations occur at the body ends. Not only are there deviations in f_s and f_b near the ends, but there exists a non-zero force in the normal direction, f_n, a result not predictable by resistive-force theory. This normal force is a finite length or end effect and arises owing to the fluid motion induced by the moving helix. Even with only one wave along the helix (figure 5*b*) Lighthill's coefficients are sufficiently accurate for many applications provided the normal force is not of interest. The Gray & Hancock coefficients are clearly a poor approximation for either case. For a helix having an elliptic longitudinal cross-section the force per unit length calculated from the Lighthill coefficients using the local cross-sectional radius, $b(s)$, is again in good agreement with slender-body theory for five waves and yielded reasonable results for $n = 1$ (figures 6*a*, *b*), except in the vicinity of the ends.

We note that Lighthill's coefficients were derived for $a/\lambda \leqslant 0{\cdot}32$ (Lighthill limits his helix pitch angle α to $\alpha^2 \leqslant 0{\cdot}2$). This is generally adequate for application in micro-organism motility; however, for larger values of a/λ the force coefficients are often unreliable. For example, for $a/\lambda = 1{\cdot}0$, $b/\lambda = 0{\cdot}01$ and $n = 5$ the zero thrust value of c/U predicted by the resistive-force theory using Lighthill's coefficients is $2{\cdot}27$ whereas the slender-body theory predicts $c/U = 3{\cdot}27$. For the case of a thrusting flagellum force coefficients capable of determining the force per unit length with similar accuracy are not presently available. In connection with flagellar hydrodynamics Johnson & Brokaw (1979) recently used the present slender-body theory to examine the accuracy of the resistive-force theory when applied to a freely swimming flagellar model generating typical finite amplitude planar waves.

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Appendix

The expansions of the integrands in (7) are constructed for field points in the vicinity of the body surface at s; consequently we let $r = \epsilon\eta$, where $\eta = O(1)$, and the distance from a point on the body centre-line at s' to the field point may be written as

$$R = (R_0^2 + \epsilon^2\eta^2 + 2\epsilon\eta \mathbf{e}_r . \mathbf{R}_0)^{\frac{1}{2}}. \quad \text{(A 1)}$$

As discussed in the introduction we limit our analysis to the consideration of bodies for which $R_0 > O(\epsilon)$ for all $s' - s > O(\epsilon)$, i.e. we exclude from our study slender bodies in which the centre-line reapproaches itself. With this restriction the outer expansion of R^{-1} (valid for points s' away from s) is

$$\left(\frac{1}{R}\right)^{\text{outer}} = \frac{1}{R_0}\left(1 - \frac{\epsilon\eta \mathbf{e}_r . \mathbf{R}_0}{R_0^2} + O(\epsilon^2)\right).$$

Writing $\mathbf{R}_0$ in terms of the unit vectors $\mathbf{e}_s$, $\mathbf{e}_n$, $\mathbf{e}_b$ at s, i.e.

$$\mathbf{R}_0 = R_{0s}\,\mathbf{e}_s + R_{0n}\,\mathbf{e}_n + R_{0b}\,\mathbf{e}_b,$$

we have

$$\left(\frac{1}{R}\right)^{\text{outer}} = \frac{1}{R_0}\left\{1 + \frac{\epsilon\eta}{R_0^2}\left[R_{0n}\cos\psi - R_{0b}\sin\psi\right] + O(\epsilon^2)\right\}. \quad \text{(A 2)}$$

Considering the integrand due to the stokeslet distribution first, we have the outer expansion of the normal velocity component (the component in the direction normal to the body centre-line at s) given by

$$u^{\text{outer}} = \mathbf{U}_S^{\text{outer}} . \mathbf{e}_n = \frac{\alpha_n(s')}{R} + (\boldsymbol{\alpha} . \mathbf{R})\frac{R_{0n} - \epsilon\eta\cos\psi}{R^3}, \quad \text{(A 3)}$$

where $\boldsymbol{\alpha} . \mathbf{R} = \boldsymbol{\alpha} . \mathbf{R}_0 - \alpha_n\,\epsilon\eta\cos\psi + \alpha_b\,\epsilon\eta\sin\psi$ and $1/R$ is given by (A 2). In the above, the vector components are expressed in terms of the unit vectors at s. Since the outer expansions for the tangential and binormal velocity components are constructed in precisely the same manner we will simply indicate what changes need to be made. The tangential component is obtained from (A 3) by replacing α_n and R_{0n} with α_s and R_{0s} and omitting the term $\epsilon\eta\cos\psi$. Similarly, the binormal component is given by (A 3) replacing α_n and $R_{0n} - \epsilon\eta\cos\psi$ with α_b and $R_{0b} + \epsilon\eta\sin\psi$.

For the inner expansion we introduce the inner or stretched variable given by $\sigma = (s' - s)/\epsilon$ and we expand R_0^2 and $\mathbf{R}$ for σ fixed and $\epsilon \to 0$, giving

$$R_0^{2\,\text{inner}} = \epsilon^2\sigma^2 - \tfrac{1}{12}\epsilon^4\sigma^4\kappa^2 + O(\epsilon^5), \quad \text{(A 4)}$$

$$\mathbf{R}^{\text{inner}} = (\epsilon\sigma r_s^{(1)} + \epsilon^3\sigma^3 r_s^{(3)})\,\mathbf{e}_s + (\epsilon r_n^{(1)} + \epsilon^2\sigma^2 r_n^{(2)} + \epsilon^3\sigma^3 r_n^{(3)})\,\mathbf{e}_n + (\epsilon r_b^{(1)} + \epsilon^3\sigma^3 r_b^{(3)})\,\mathbf{e}_b + O(\epsilon^4), \quad \text{(A 5)}$$

where $\mathbf{e}_s$, $\mathbf{e}_b$, $\mathbf{e}_n$ are the unit vectors at s and

$$r_s^{(1)} = -1, \quad r_s^{(3)} = \tfrac{1}{6}\kappa^2,$$

$$r_n^{(1)} = -\eta\cos\psi, \quad r_n^{(2)} = -\tfrac{1}{2}\kappa, \quad r_n^{(3)} = -\frac{1}{6}\frac{\partial\kappa}{\partial s},$$

$$r_b^{(1)} = \eta\sin\psi, \quad r_b^{(3)} = \tfrac{1}{6}\kappa\tau,$$

τ being the torsion, i.e. the measure of the rate at which the centre-line $\mathbf{x}_0(s,t)$ twists out of the osculating plane ($\partial\mathbf{e}_b/\partial s = \tau\mathbf{e}_n$). Using (A 4) and (A 5) we find the inner expansion of R^{-1} as

$$\left(\frac{1}{R}\right)^{\text{inner}} = \frac{1}{\epsilon\Delta}\left\{1+\frac{\epsilon\sigma^2}{\Delta^2}R_1+\frac{\epsilon^2}{\Delta^2}(\sigma^4R_2+\sigma^3R_3)+O(\epsilon^3)\right\}. \quad \text{(A 6)}$$

where

$$R_1 = -\tfrac{1}{2}\eta\kappa\cos\psi, \quad R_2 = \frac{\kappa^2}{24}\left(1+\frac{9\eta^2\cos^2\psi}{\Delta^2}\right),$$

$$R_3 = -\tfrac{1}{6}\eta\left(\frac{\partial\kappa}{\partial s}\cos\psi+\kappa\tau\sin\psi\right),$$

$$\Delta = (\sigma^2+\eta^2)^{\frac{1}{2}}.$$

The stokeslet strength has the inner expansion about $s' = s$

$$\boldsymbol{\alpha}^{\text{inner}}(s',t) = \boldsymbol{\alpha}^{(0)}+\epsilon\sigma\boldsymbol{\alpha}^{(1)}+\epsilon^2\sigma^2\boldsymbol{\alpha}^{(2)}+O(\epsilon^3), \quad \text{(A 7)}$$

$$\boldsymbol{\alpha}^{(0)} = \alpha_s(s,t)\,\mathbf{e}_s+\alpha_n(s,t)\,\mathbf{e}_n+\alpha_b(s,t)\,\mathbf{e}_b,$$

$$\boldsymbol{\alpha}^{(1)} = \left(\frac{\partial\alpha_s}{\partial s}-\kappa\alpha_n\right)\mathbf{e}_s+\left(\frac{\partial\alpha_n}{\partial s}+\kappa\alpha_s+\tau\alpha_b\right)\mathbf{e}_n+\left(\frac{\partial\alpha_b}{\partial s}-\tau\alpha_n\right)\mathbf{e}_b, \quad \text{(A 8)}$$

where $(\mathbf{e}_s, \mathbf{e}_n, \mathbf{e}_b)$ are the unit vectors at s, and α_ν ($\nu = n,s,b$) are the components of $\boldsymbol{\alpha}$ at s in terms of these unit vectors. The detailed expression for $\alpha^{(2)}$ is not presented since it is rather lengthy and plays no essential role to the order at which we are working. Using (A 5), (A 7) and (A 8) the inner expansion of the normal velocity component due to the stokeslet distribution is found to be

$$u^{\text{inner}} = \mathbf{U}_S^{\text{inner}}.\,\mathbf{e}_n = (\alpha_n^{(0)}+\epsilon\sigma\alpha_n^{(1)}+\epsilon^2\sigma^2\alpha_n^{(2)})/R$$
$$+(\boldsymbol{\alpha}.\boldsymbol{R})\,(\epsilon r_n^{(1)}+\epsilon^2\sigma^2 r_n^{(2)}+\epsilon^3\sigma^3 r_n^{(3)})/R^3+O(\epsilon^2), \quad \text{(A 9)}$$

where $1/R$ is given by its inner expansion (A 6) and

$$\boldsymbol{\alpha}.\mathbf{R} \simeq \epsilon(\sigma C_{11}+C_{10})+\epsilon^2(\sigma^2C_{22}+\sigma C_{21})$$
$$+\epsilon^3(\sigma^3C_{33}+\sigma^2C_{32})+O(\epsilon^4),$$

$$C_{11} = \alpha_s^{(0)}r_s^{(1)}, \quad C_{10} = \alpha_n^{(0)}r_n^{(1)}+\alpha_b^{(0)}r_b^{(1)},$$

$$C_{22} = \alpha_s^{(1)}r_s^{(1)}+\alpha_n^{(0)}r_n^{(2)}, \quad C_{21} = \alpha_n^{(1)}r_n^{(1)}+\alpha_b^{(1)}r_b^{(1)},$$

$$C_{33} = \alpha_s^{(2)}r_s^{(1)}+\alpha_n^{(1)}r_n^{(2)}+\alpha_s^{(0)}r_s^{(3)}+\alpha_n^{(0)}r_n^{(3)}+\alpha_b^{(0)}r_b^{(3)}$$

$$C_{32} = \alpha_n^{(2)}r_n^{(1)}+\alpha_b^{(2)}r_b^{(1)}.$$

The inner expansion for the tangential component is readily obtained from (A 9) by replacing $\alpha_n^{(k)}$, $r_n^{(1)}$, $r_n^{(2)}$, $r_n^{(3)}$ with $\alpha_s^{(k)}$, $\sigma r_s^{(1)}$, 0, $r_s^{(3)}$ respectively. For the binormal component $\alpha_n^{(k)}$, $r_n^{(1)}$, $r_n^{(2)}$, $r_n^{(3)}$ is replaced in (A 9) by $\alpha_n^{(k)}$, $r_b^{(1)}$, 0, $r_b^{(3)}$.

The common-part expansion is most easily obtained by taking the inner limit of the outer expansion. For the normal velocity component this amounts to substituting into (A 3) and (A 2) the inner expansions for α, R_0^{-1} and $R_{0\nu}$ ($\nu = n,s,b$) which can be obtained from (A 7), (A 4) and (A 5) respectively. The common-part expansion for the tangential and binormal components is similarly constructed. The uniformly valid expansion is, as discussed in § 3, the sum of the outer and inner expansions minus the common part.

Now, considering the doublet distribution, we observe that the outer expansion is of order β/R_0^3 where β is the non-dimensional strength and $R_0 = O(1)$ for the outer region. Based on what has been found for the slender torus and straight body we assume *a priori* that $O(\beta) = O(\epsilon^2\alpha)$. We then conclude that the contribution to the velocity field from the outer expansion of the doublet distribution is of higher order than those terms retained and therefore can be neglected. This assumption on the order of the doublet strength will of course be verified later when the doublet strength is determined. Furthermore, the common-part expansion is readily obtained from the inner limit of the outer expansion and is therefore also of order $\epsilon^2\alpha$. This can be checked by considering the outer limit of the inner expansion, which is a more formidable task. Therefore we need only calculate the inner expansion of the doublet distribution. For the inner expansion we could expand the doublet strength about $s' = s$ as was done for the stokeslet; however, that particular choice makes the analysis unnecessarily complicated. A better choice is motivated by noting that for a straight prolate spheroid the doublet has a parabolic distribution between the foci of the body (Chwang & Wu 1975). This suggests that we take

$$\boldsymbol{\beta}(s,t) = \epsilon^2\mathbf{B}(s,t)\,(s_1^2-s^2). \tag{A 10}$$

The doublet strength is then expanded about $s' = s$ as follows:

$$\boldsymbol{\beta}^{\text{inner}}(s',t) = \boldsymbol{\beta}^{(0)}+\epsilon\sigma\boldsymbol{\beta}^{(1)}+\epsilon^2\sigma^2\boldsymbol{\beta}^{(2)}+O(\epsilon^3), \tag{A 11}$$

where $\boldsymbol{\beta}^{(k)} = \epsilon^2\mathbf{B}^{(k)}(s,t)\;(s_1^2-s'^2)$ for $k = 0,1,2,\ldots$, with $\mathbf{B}^{(k)}$ given by (A 8) replacing α_ν ($\nu = n,s,b$) by B_ν. What we have done here is to expand the slowly varying portion of the doublet strength $\mathbf{B}$ about $s'=s$ while leaving the parabolic modulation intact, thereby properly accounting for the behaviour of the doublet strength near the body ends. This procedure is in direct analogy with the two-timing or two-variable expansion method found in perturbation theory, in which we have a fast and a slow variable (Van Dyke 1975). The inner expansion of the integrand due to the doublet distribution is then easily obtained from the inner expansion of the stokeslet distribution (equation (A 9) for the normal velocity component) by replacing $\alpha_\nu^{(k)}$ ($\nu = n,s,b$), $1/R$ and $1/R^3$ with $\beta_\nu^{(k)}$, $1/R^3$ and $-3/R^5$ respectively.

Our first approximation to the velocity field [equation (7)] thus becomes

$$\mathbf{u}(s,r,\psi) = \int_{-\sigma_1}^{\sigma_2}(\mathbf{U}_S+\mathbf{U}_D)^{\text{inner}}\epsilon\,d\sigma+\int_{-s_1}^{s_1}(\mathbf{U}_S^{\text{outer}}-\mathbf{U}_S^{\text{common part}})\,ds'+O(\epsilon^2\alpha), \tag{A 12}$$

where $\sigma_1 = (s_1+s)/\epsilon$ and $\sigma_2 = (s_1-s)/\epsilon$. Since the unknown singularity strengths have been expanded about $s' = s$ in the inner and common-part expansions it is now possible to integrate many of the terms in (A 12). After integration and evaluation of the velocity on the body surface neglecting terms of $O(\alpha\epsilon^2\ln\epsilon)$ we find (10), (11) and (12) (see §3).

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